Example 5-3: Impulse Response of Cascade

To illustrate the utility of the results that we have obtained for cascaded LTI systems, consider the cascade of two systems defined by

\[
 h_1[n] = \begin{cases} 
 1 & 0 \leq n \leq 3 \\
 0 & \text{otherwise} 
\end{cases} \quad h_2[n] = \begin{cases} 
 1 & 1 \leq n \leq 3 \\
 0 & \text{otherwise} 
\end{cases}
\]

The results of this section show that the overall cascade system has impulse response

\[
 h[n] = h_1[n] * h_2[n]
\]

Therefore, to find the overall impulse response we must convolve \( h_1[n] \) with \( h_2[n] \). This can be done by using the polynomial multiplication algorithm of Section 5-3.3.1. In this case, the computation is as follows:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( n &lt; 0 )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>( n &gt; 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_1[n] )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( h_2[n] )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( h_1[0]h_2[n] )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( h_1[1]h_2[n-1] )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( h_1[2]h_2[n-2] )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( h_1[3]h_2[n-3] )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( h[n] )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Therefore, the equivalent impulse response is

\[
 h[n] = \sum_{k=0}^{6} b_k \delta[n - k]
\]

where \( \{b_k\} \) is the sequence \( \{0, 1, 2, 3, 3, 2, 1\} \). This result means that a system with this impulse response \( h[n] \) can be implemented either by the single difference equation

\[
 y[n] = \sum_{k=0}^{6} b_k x[n - k]
\]

where \( \{b_k\} \) is the above sequence, or by the pair of difference equations

\[
 w[n] = \sum_{k=0}^{3} x[n - k] \quad y[n] = \sum_{k=1}^{3} w[n - k]
\]