The Basic Geometry Behind A Camera Lens And A Magnifying Glass

by

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THE BASIC GEOMETRY BEHIND A CAMERA LENS

Lenses used in cameras are called converging lenses because they bend rays of light that are parallel to the normal line through the center plane of the lens. As an example of how a converging lens bends horizontal rays of light, see *figure 1* below. As the horizontal light rays pass through the lens they all end up going through the same point on the other side of the lens. The lens causes all horizontal light rays to converge at one point which is known as a focal point.

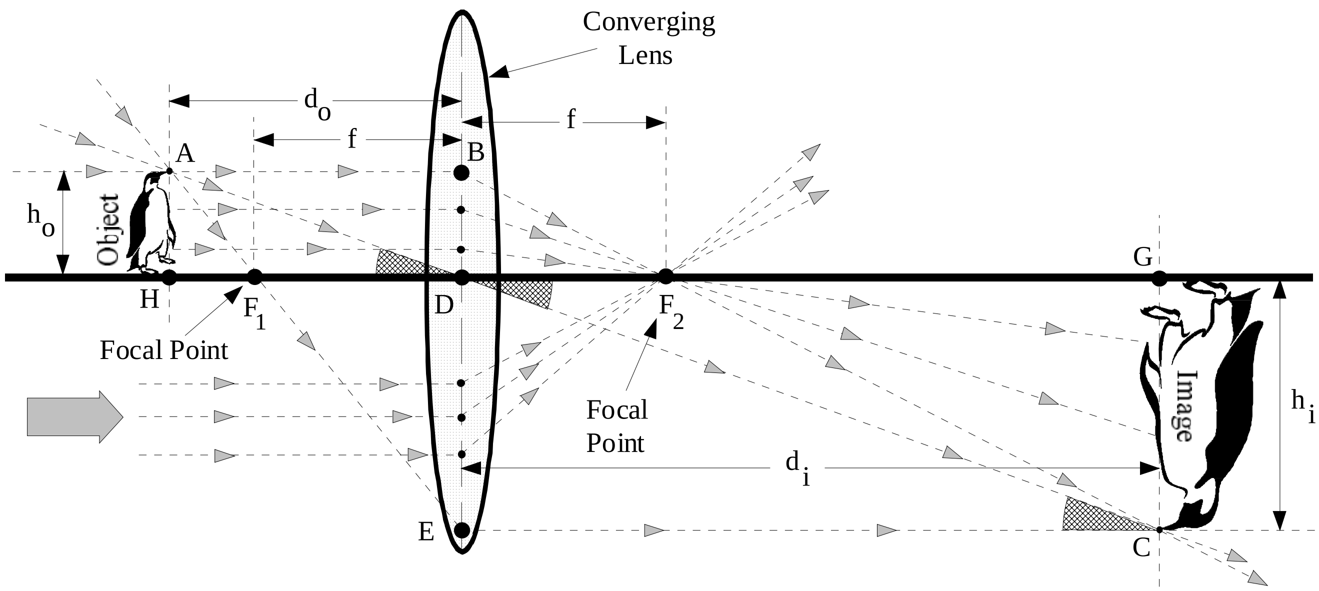

Figure 1. A converging lens. Horizontal light rays to the left of the lens are bent by the lens and all go through the focal point F_2 on the right.

Each lens actually has two focal points which lie on opposite sides of the lens, but they are equidistant from the center of the lens. In the above figure the two focal points are denoted by F_1 and F_2. We assume the object being viewed through the lens is the penguin on the left. The image formed is shown to the right of the lens and appears inverted. Note the image on the right appears larger than the original object which is on the left. Both the object and the image lie outside the vertical band or region which is between the two focal points F_1 and F_2.

The entire action of the lens may be understood by tracing rays of light. In fact, only two special rays need to be traced to determine both the size and the position of the image. First, consider the line starting at point A and passing through point B. Since this line is parallel to the normal line through the lens center, the ray of light bends as it passes through the lens at point B in such a manner that it goes through the second focal point labeled as F_2. In fact, all rays of light parallel to the center line $\overline{F_1F_2}$ will pass through the focal point F_2. For example, trace the three rays of light that start in the lower-left part of *figure 1*. Any ray of light not parallel to the center line $\overline{F_1F_2}$ will not pass through a focal point.

The line $\overline{AD}$ in *figure 1* above is a special line which goes through the exact center of the lens as indicated by point D. The ray of light represented by the line $\overline{AD}$ is the only one shown in *figure 1* which is not bent at all by the lens. In fact, only lines through D will not be bent by the lens. Thus points A, D, and C in the above figure are collinear. The size and position of the image are determined by the two rays of light represented by the paths $\overline{AB}\,\overline{F_2C}$ and $\overline{ADC}$. Point C is determined as the point of intersection of the two lines $\overline{BF_2}$ and $\overline{AD}$.

THE THIN-LENS EQUATION

In *figure 1* above we have labeled the focal length of the lens using the letter f. Thus f is the same distance the two focal points lie from the lens center. We have also labeled the distance the object and image lie from the lens center by using the notations d_o and d_i, respectively, for the object and image. Next we will derive the fundamental equation which gives the relationship between the three quantities f, d_o, and d_i.

In *figure 1*, $\angle ADH$ has the same measure as $\angle GDC$ and $\angle DCE$. $\triangle ADH$ is similar to $\triangle DCE$. Thus we have the following relationship between the object and image heights and the object and image distances from the lens center. We let h_o and h_i denote the respective heights of the object and image.

$$\frac{h_o}{d_o} = \frac{h_i}{d_i} \qquad \text{or} \qquad \frac{h_i}{h_o} = \frac{d_i}{d_o}$$

If we place an xy-coordinate system with its origin at point D then the line $\overline{BF_2}$ would have slope $\frac{-h_o}{f}$ and go through the point $F_2(f, 0)$. The equation of the line is:

$$y = \frac{-h_o}{f}\,(\,x - f\,)$$

Now using these equations and substituting the coordinates of point C, $y = -h_i$ when $x = d_i$, we can derive one of the fundamental equations for the lens:

$$-h_i = \frac{-h_o}{f}(\,d_i - f\,) \qquad \text{Substitute } y = -h_i \text{ and } x = d_i.$$

$$-h_i = \frac{-h_o \cdot d_i}{f} + h_o \qquad \text{Multiply out the right side.}$$

$$\frac{-h_i}{h_o} = \frac{-d_i}{f} + 1 \qquad \text{Divide both sides by } h_o.$$

$$\frac{-d_i}{d_o} = \frac{-d_i}{f} + 1 \qquad \text{Substitute } \frac{h_i}{h_o} = \frac{d_i}{d_o}.$$

$$\frac{-1}{d_o} = \frac{-1}{f} + \frac{1}{d_i}$$ Divide both sides by d_i.

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}$$ Add $\frac{1}{f} + \frac{1}{d_o}$ to both sides.

This last equation is known as the thin-lens equation.

THE THIN-LENS EQUATION

$$\boldsymbol{\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}}$$ where

$\boldsymbol{f}$ = the focal length of the lens
$\boldsymbol{d_o}$ = the distance between the center of the lens and the object.
$\boldsymbol{d_i}$ = the distance between the center of the lens and the image.

Next, refer to *figure 2* below. This figure differs from *figure 1* in that the penguins are not standing on the center line. In fact, the center line partially goes through each penguin. We also assume the object being photographed is taller than the physical height of the camera lens or the physical size of one picture frame on the film.

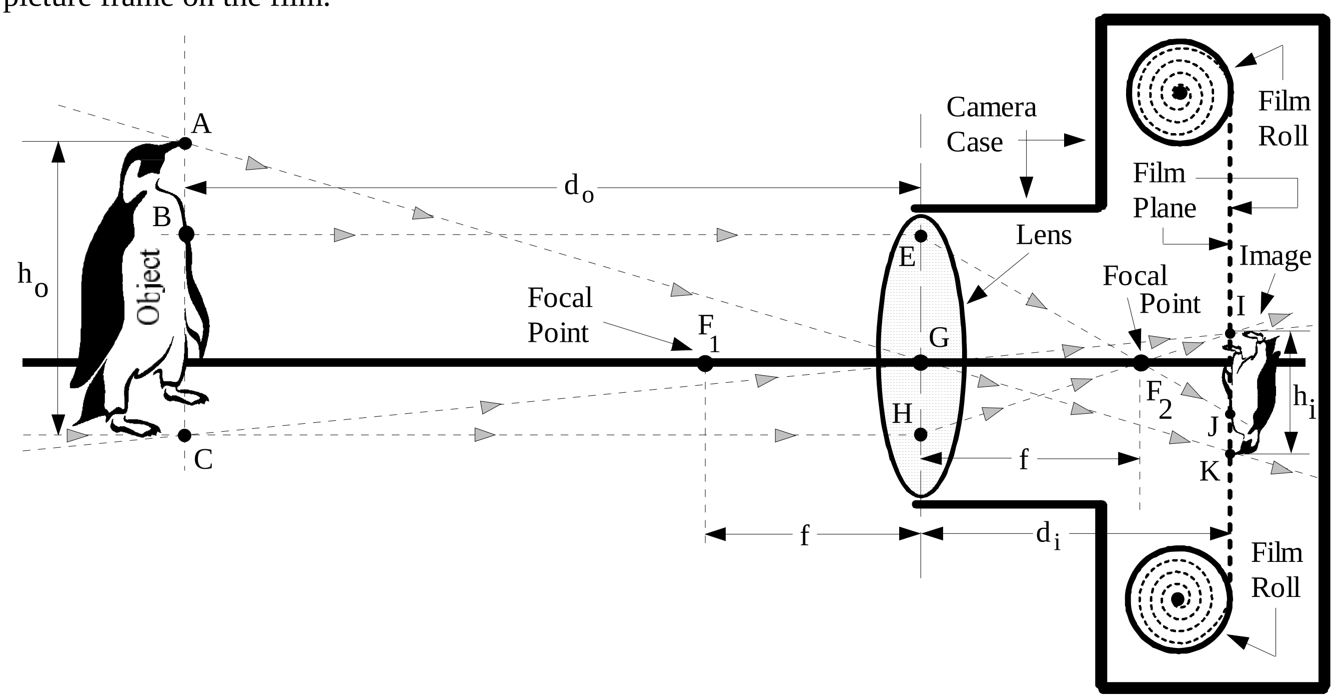

Figure 2. A camera image of an object is focused on the film plane behind the camera lens.

The Thin-Lens equation can be expressed in a somewhat simpler form by considering all distances as being measured in terms of multiples of the focal length f. If we let $z \cdot f = d_o$ and let $y \cdot f = d_i$ then z and y measure the object and image distances in terms of multiples of the focal length. Thus in *figure 2* above, z measures the length d_o and y measures the length d_i, but the units for z and y are both in terms of the focal length. Substituting these values in the thin-lens equation produces a simple relationship between y and z.

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i} \qquad \text{The thin-lens equation}$$

$$\frac{1}{f} = \frac{1}{z \cdot f} + \frac{1}{y \cdot f} \qquad \text{Substitute } d_o = z \cdot f \text{ and } d_i = y \cdot f.$$

$$1 = \frac{1}{z} + \frac{1}{y} \qquad \text{Multiply through by } f.$$

This last equation can be expressed in an even simpler form by introducing two new measurements called *a* and *b* shown in *figure 3* below. The quantities *a* and *b* are also measured in terms of multiples of the focal length f. In *figure 3* below we assume $a \cdot f = d_o - f$ and $b \cdot f = d_i - f$. It is best to think that a measures the length $d_o - f$, but the units of a are in terms of focal lengths. Also, we can think that b measures the length of $d_i - f$, where the units of b are also in terms of focal lengths.

While looking at *figure 3* below you can assume a measures the number of focal lengths the object lies to the left of the left focal point F_1. You can also assume b measures the number of focal lengths the image lies to the right of the right focal point F_2. Recall that the object and the image lie outside the region between the two vertical lines through the two focal points.

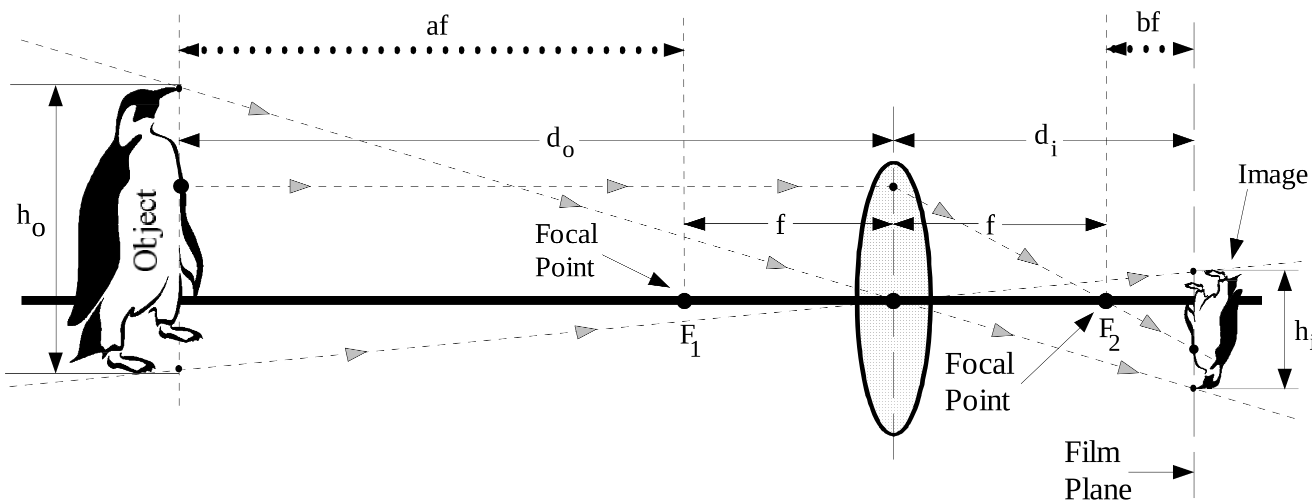

Figure 3. *The distances of the object and image beyond one focal length as represented by af and bf.*

Now we may derive the simple relationship that exists between the two numbers a and b.

$$a \cdot f \; + \; f \; = \; d_o = z \cdot f \;\text{ and }\; b \cdot f \; + \; f \; = \; d_i = y \cdot f \qquad \text{See } \textit{figure 3} \text{ above.}$$

$$(\,a+1\,)\,f \; = \; z \cdot f \;\text{ and }\; (\,b+1\,)\,f = y \cdot f \qquad \text{Factor out } f.$$

$$a+1 \; = \; z \;\text{ and }\; b+1 = y \qquad \text{Divide by } f.$$

$$1 \; = \; \frac{1}{z} \; + \; \frac{1}{y} \qquad \text{Previous thin-lens equation.}$$

$$1 \; = \; \frac{1}{a+1} \; + \; \frac{1}{b+1} \qquad \text{Substitute } z = a+1 \text{ and } y = b+1.$$

$$(\,a+1\,)(\,b+1\,) \; = \; (\,b+1\,) \; + \; (\,a+1\,) \qquad \text{Clear the denominators.}$$

$$a\,b \; + \; a \; + \; b \; + \; 1 \; = \; a \; + \; b \; + \; 1 \; + \; 1 \qquad \text{Simplify.}$$

$$a\,b \; = \; 1 \qquad \text{Subtract } a+b+1 \text{ from both sides.}$$

$$a \; = \; \frac{1}{b} \qquad \text{Divide by } b.$$

This last equation has a simple but important and eminently useful interpretation. a is the number of focal lengths the object lies to the left of the left focal point. b is the number of focal lengths the image lies to the right of the right focal point. Thus interpreting the last equation we know the further the object lies away from the left focal point the closer the image lies near the right focal point. Vice versa, the further the image lies to the right of the right focal point the closer the object must be to the left focal point. The values a and b are reciprocals of each other.

Note that if $a = 1$ then we also have $b = 1$ which means when the image lies two focal lengths to the left of the lens center, the image also lies two focal lengths to the right of the lens center. In this case, and only in this case, is it true that the object and image heights are the same. In fact, next we will show that the magnification factor of the object is the number $\frac{1}{a}$ which is the same as the number b.

Given the proportion via similar triangles that $\frac{h_o}{d_o} = \frac{h_i}{d_i}$ we can solve for h_i.

$$h_i \; = \; \frac{d_i}{d_o} \cdot h_o$$

This shows the fraction $\frac{d_i}{d_o}$ is the magnification factor which determines the image height from the object height.

Next we show the magnification factor $\frac{d_i}{d_o}$ is the same as the number b.

$$\frac{d_i}{d_o} = \frac{b \cdot f + f}{a \cdot f + f} = \frac{(b+1) \cdot f}{(a+1) \cdot f} = \frac{b+1}{a+1} = \frac{b+1}{\frac{1}{b}+1} = \frac{(b+1) \cdot b}{(1+b)} = b$$

So the number b can be used to determine the image height from the object height.

$$h_i = b \cdot h_o$$

The interplay between the numbers a and b can be illustrated as follows. When $a = 3$, the object is 3 focal lengths to the left of the left focal point (it is 4 focal lengths from the lens center) and the image is $\frac{1}{3}$ the size of the object and the image lies $\frac{1}{3}$ of a focal length to the right of the right focal point (the image is $1 + \frac{1}{3} = \frac{4}{3}$ focal lengths away from the lens center).

In normal practice the object is usually much more than 1 focal length to the left of the left focal point. In this case $a > 1$ which implies $b < 1$. Thus when a camera lens focuses the image on the film, the lens never has to move more than one focal length to bring the image into focus. The image is focused by moving the lens relative to the plane of the film inside the camera. See *figure 2* which is an example of the normal photographic situation where $a > 1$ and $b < 1$.

When doing close-up photography the object will be between 1 and 2 focal lengths from the lens center and in this case a will satisfy the inequality $0 < a < 1$ which in turn implies $b > 1$. Since b is the same as the magnification factor, the image produced on the film will be larger than the real-life image size. Thus relatively small objects can be blown up to sizes larger than real-life. For example, imagine photographing the head of a pin. *Figure 1* corresponds to close-up photography.

Referring back to *figure 1*, we can imagine how the image size and position vary as the object position varies. Think of the line $\overline{AD}$ (D is the lens center) as swiveling or pivoting about point D. Point A remains at the peak of the left penguin's bill and in fact remains at the same height above the center line no matter where the left penguin (the object) is positioned. When point A is far away from the lens center the line $\overline{AD}$ becomes more horizontal and the image height decreases. As point A moves closer towards the left focal point F_1 (but A remains to the left of F_1) the line $\overline{AD}$ becomes more vertical and the image height increases. In fact, $\angle DBF_2$ remains constant regardless of the distance the object is placed from the lens. $\tan(\angle DBF_2) = \frac{f}{h_o}$ which is independent of d_o.

When point A is between 1 and 2 focal lengths from the lens center the image size will actually be larger than the object size. Thus A must be positioned between 1 and 2 focal lengths in order to do close-up photography. In practice, point A would never get closer to the lens center than 1 focal length.

If you look closely at *figure 2* you can determine that points A, B, and C on the object penguin correspond one-for-one with the points K, J, and I on the image penguin. Points A, B, and C are vertically aligned on the object while points K, J and I are vertically aligned on the image. In *figure 2* we assume these three points all lie in the film plane.

In particular, you should try ray tracing point B on the object penguin's breast. The horizontal line goes through the lens at point E, travels down through the focal point F_2, and finally reaches point J on the image penguin's breast. Points B and J are corresponding points on the two penguins. Point C between the object penguin's feet ray traces to point I which is between the image penguin's feet. Point A at the tip of the object penguin's bill maps directly through the center of the lens to point K which is the tip on the image penguin's bill. If the film in our camera were taking $35\,mm$ slides then the image height h_i would have to be sufficiently small to fit within one slide-shot picture frame on the film.

If you use a ruler and compute ratios you can determine the different values for a and b that are in both *figure 1* and *figure 3*. *Figure 2* is essentially the same as *figure 3*. In fact, the reason for making *figure 1* and *figure 3* different is so you could visualize the difference between the *a* value being less than one focal length (*figure 1*) and the a value being greater than one focal length (*figure 2* and *figure 3*).

For *figure 1* we estimate the actual focal length to be about $24\,mm$. The distance between H and F_1 is about $10\,mm$ so the a value should be $\frac{10}{24} = \frac{5}{12}$. This means the b value should be the reciprocal, $\frac{12}{5} = 2.4$. Since the b value is the magnification factor, we assume the image is about 2.4 times as large as the object. In fact, when we measure the heights of the object and image we find their heights are 12 mm and $29\,mm$ respectively. Note that $2.4 \times 12 = 28.8$, which rounds to 29 to the nearest mm. In *figure 1*, the distance between F_2 and G is about $58\,mm$. Note that with $f = 24$, we should find the length of $\overline{F_2G}$ will be $24 \times 2.4 = 57.6$ which rounds up to 58 to the nearest mm.

For *figure 3* we estimate the actual focal length to be about $24\,mm$. The distance between the object and F_1 is about $58\,mm$ so the a value should be $\frac{58}{24} = \frac{29}{12}$. This means the b value should be the reciprocal, $\frac{12}{29} \approx 0.4138$ or about 41%. The height of the object is about $33\,mm$ while the height of the image is just under $14\,mm$. Note that $33 \times 0.4138 \approx 13.65$, which would round up to about 14 to the nearest mm. The distance between the image and F_2 in *figure 3* is about $10\,mm$. Note that $24 \times 0.4138 \approx 9.93$, or rounded to the nearest mm, about $10\,mm$.

So the values predicted by the equations actually match what we can measure in the figures. Understanding the fundamental relationship between the a and b values allows you to interpret the basic geometry behind a camera lens. This may not make you a better photographer, nor will it qualify you to become a lens designer, but you should now have a better understanding of the mathematics behind a camera lens, and that by itself is progress!

THE BASIC GEOMETRY BEHIND A MAGNIFYING GLASS

Next we consider the geometry of a magnifying glass. With a magnifying glass the viewer's eye is considered to be fixed at one focal point and the object being viewed is on the other side of the lens, but within a distance shorter than the focal length. The image appears to the left of the left focal point, appears enlarged, and has the same orientation as the original object. The image is a virtual one. This means the image position appears to the eye as if it were actually at the place shown in *figure 4* below, even though no light rays make the image there. The true image is formed by the light rays that enter the viewer's eye.

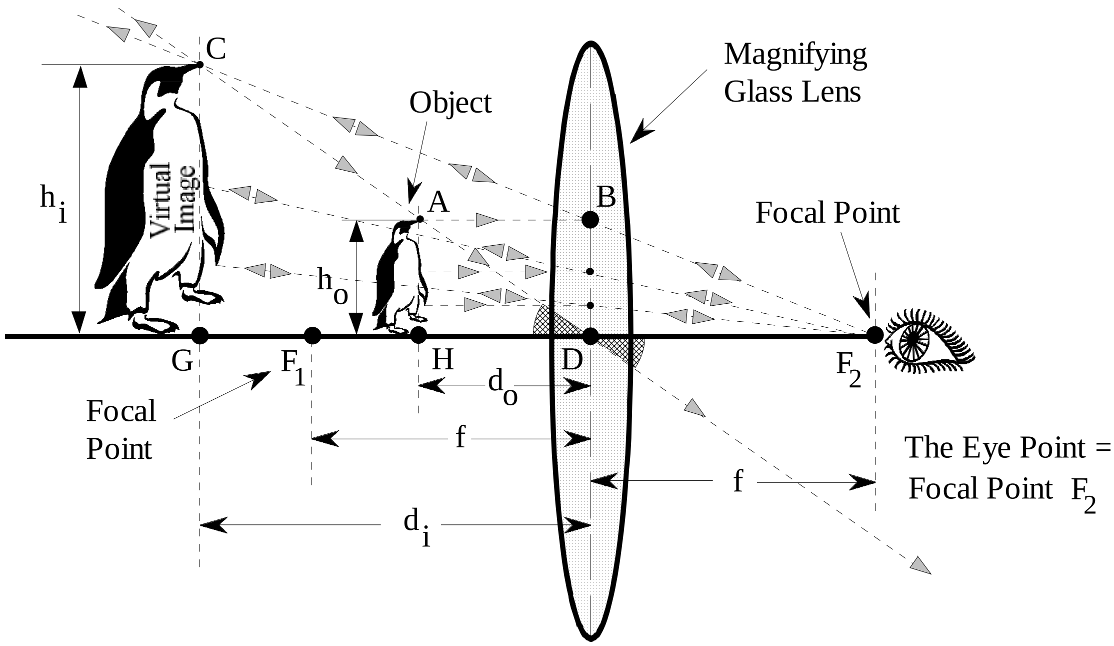

Figure 4. The virtual image of an object under a magnifying glass.

As with a camera lens, horizontal rays of light may be traced from the object, through the lens and through the focal point F_2. The difference is that these same virtual rays may be traced backwards from F_2 along straight lines until they intersect the vertical line $\overline{CG}$. The line $\overline{CD}$ through the center of the lens is not bent at all by the lens. The ray of light that extends from the large penguin's bill at point C to the focal point F_2 is straight. In fact, all virtual rays from the virtual image penguin that aim directly at the focal point F_2 are not bent by the lens.

Point C in *figure 4* is the point of intersection of the two lines $\overline{BF_2}$ and $\overline{AD}$. $\triangle AHD$ is similar to $\triangle CGD$ and we have the same ratio between the object and image heights and the object and image distances from the lens, as with the converging camera lens.

$$\frac{h_o}{d_o} = \frac{h_i}{d_i}$$

If we place an xy-coordinate system with its origin at point D then the equation of the line $\overline{BF_2}$ would be the same as for the converging lens.

$$y = \frac{-h_o}{f}(x - f)$$

The difference is that we now substitute $y = h_i$ when $x = -d_i$ (the signs of both coordinates of point C are reversed from what they were for the camera) and we derive a new fundamental equation for a magnifying glass.

$$h_i = \frac{-h_o}{f}(-d_i - f)$$ Substitute $y = h_i$ and $x = -d_i$.

$$h_i = \frac{h_o d_i}{f} + h_o$$ Multiply out the right side.

$$\frac{h_i}{h_o} = \frac{d_i}{f} + 1$$ Divide both sides by h_o.

$$\frac{d_i}{d_o} = \frac{d_i}{f} + 1$$ Substitute $\frac{d_i}{d_o}$ for $\frac{h_i}{h_o}$.

$$\frac{1}{d_o} = \frac{1}{f} + \frac{1}{d_i}$$ Divide both sides by d_i.

$$\frac{1}{f} = \frac{1}{d_o} - \frac{1}{d_i}$$ Subtract $\frac{1}{d_i}$ from both sides.

THE FUNDAMENTAL EQUATION FOR A MAGNIFYING GLASS

$$\frac{1}{f} = \frac{1}{d_o} - \frac{1}{d_i}$$

Note that this is analogous to, but different from, the thin-lens equation for a camera.

Analogous to what was done with the converging camera lens, we can let the quantity b measure the distance between the image (virtual) and the focal point F_2 and we can let a measure the distance between the object and the focal point F_1. The units for a and b are in terms of the focal length f. See *figure 5* below.

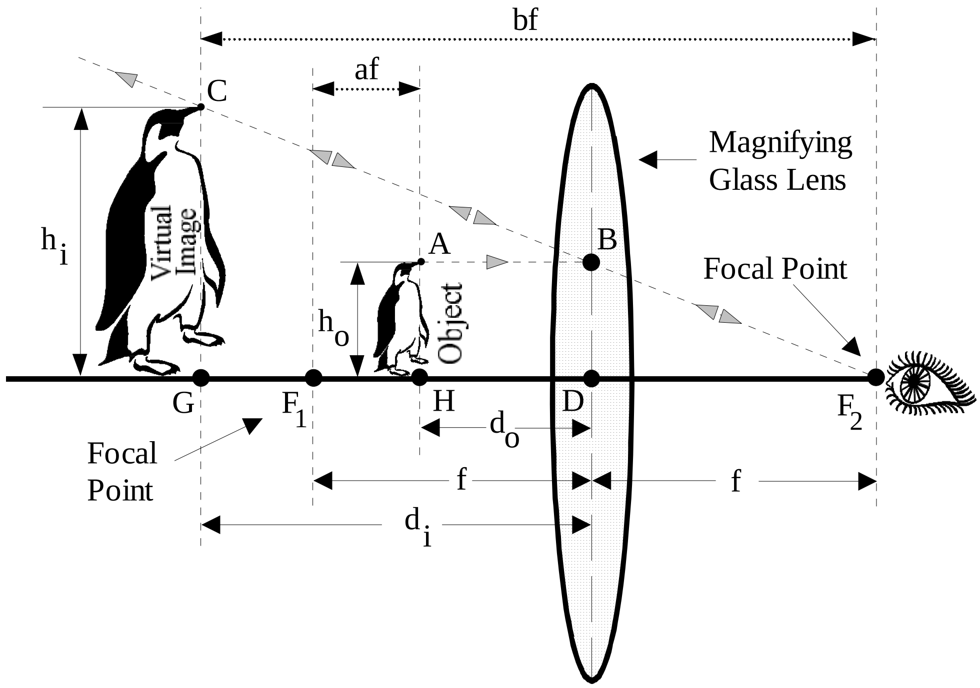

Figure 5. Focal length measurements bf and af relative to the image and the object and the two focal points.

$$\frac{1}{f} = \frac{1}{d_o} - \frac{1}{d_i}$$ Fundamental equation for a magnifying glass.

$$\frac{1}{f} = \frac{1}{f - af} - \frac{1}{bf - f}$$ Substitute $d_o = f - af$ and $d_i = bf - f$.

$$1 = \frac{1}{1-a} - \frac{1}{b-1}$$ Multiply through by f.

$$(1-a)(b-1) = (b-1) - (1-a)$$ Multiply through by $(1-a)(b-1)$.

$$-ab + a + b - 1 = b - 1 - 1 + a$$ Expand left side; simplify right side.

$$-ab = -1$$ Add $-a - b + 1$ to both sides.

$$a = \frac{1}{b}$$ Divide both sides by $-b$.

Note that this equation is the same as for a regular camera lens. The smaller the value of a the larger the virtual image. Also, the larger the value of a the smaller the virtual image.

As was also true for a camera lens, the number $b = \frac{1}{a}$ is the magnification factor between the object and the image. By looking at *figure 5* we can see that the magnification factor is $\frac{h_i}{h_o}$ and next we show this fraction is the same as b.

$$\frac{h_i}{h_o} = \frac{d_i}{d_o} = \frac{bf - f}{f - af} = \frac{f(b-1)}{f(1-a)} = \frac{b-1}{1-a} = \frac{b-1}{1-\frac{1}{b}} = \frac{(b-1)b}{(b-1)} = b$$

In the case of a magnifying glass the value b satisfies the inequality: $b > 2$. This implies that $0 < a < \frac{1}{2}$.